Academic Year of 2014
Admission to the Master's Program
Department of Intelligence Science and Technology
Graduate School of Informatics, Kyoto University
(Fundamentals of Informatics)
(International Course)

13:30 - 15:00, February 12, 2014

NOTES

1. This is the Question Booklet in 3 pages including this front cover.
2. Do not open the booklet until you are instructed to start.
3. After start, check the number of pages and notify proctors (professors) immediately if you find missing pages or unclear printings.
4. This booklet has 2 questions written in English. **Solve all questions.**
5. Write your answers in English, unless specified otherwise.
6. Read carefully the notes on the Answer Sheets as well.

Q. 1 Let $G = (V, E)$ be an undirected graph, where V denotes a set of vertices whose labels are indexed by integers $(0, \ldots, n)$, and E denotes a set of edges. Consider the following pseudo code function of a graph traversal algorithm in (a), where i, j are indices of vertices, *visited*[j] is an element of an array, which states if the corresponding vertex has been visited or not, all the elements are initialized as false. For a certain graph G', $abc(0)$ starting from vertex 0 has a following trace in (b).

Pseudo Code:

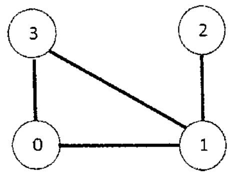
```
Void abc(vertex_id i)
set visited[i] to true;
for each vertex j adjacent to the vertex i in the graph
        if visited[j] is false
                abc(j);
        else
                print "check j"
print "j done"
```

(a)

```
abc(0)
     abc(3)
          check 0
          abc(1)
               check 3
               abc(2)
                    check 1
                    check 0
               2 done
               check 0
          1 done
     3 done
     check 2
     check 1
0 done
```

(b)

1.1 Draw the graph G'.
1.2 Function *abc* implements a well-known traversal algorithm. What is its name?
1.3 Provide adjacency sets in G'.
1.4 Provide the trace of $abc(3)$ for the following graph. Assume that adjacent vertices in each adjacency set are listed in ascending order.



Q. 2 In computing, Sign-Magnitude, 1's complement and 2's complement are used to represent positive/negative numbers in binary number systems.

2.1 Convert two decimal numbers 26 and -117 to binary using 8-bit Sign-Magnitude, 1's complement and 2's complement representations, respectively.
2.2 What disadvantages does Sign-Magnitude representation have? What is the advantage of 2's complement representation over 1's complement?
2.3 What is the procedure for converting a decimal number to 2's complement representation using 1's complement?
2.4 Provide another idea of converting a decimal number to 2's complement representation without using 1's complement. Please use -117 as an example.

Q. 1 The entropy of a random variable X taking values in a finite set $\mathcal{X}$ is a function of the probabilities $\{P(X = x), x \in \mathcal{X}\}$, defined as $H(X) := -\sum_{x\in\mathcal{X}} P(X = x) \log P(X = x)$. We consider in the following such a finite set $\mathcal{X}$ as well as two random variables X, Y taking values in that set.

1. Describe in one sentence what the entropy $H(X)$ measures. Provide examples of random variables with low and high entropy.
2. What is the definition of the conditional entropy $H(X|Y)$? Provide a formula that uses the conditional probabilities $P(X = x|Y = y)$ and probabilities $P(Y = y)$ for $x, y \in \mathcal{X}$. Show that $H(X|Y) \geq 0$.
3. We write $H(X, Y)$ for the entropy of (X, Y), the joint random variable formed by X and Y that takes values in the finite set $\mathcal{X} \times \mathcal{X}$. Prove the chain rule formula $H(X, Y) = H(X|Y) + H(Y)$.
4. A twice differentiable real-valued function defined on an interval of $\mathbb{R}$ is convex if its second order derivative is non-negative everywhere on that interval. Prove that the function f defined as $f(x) = -\log(x)$ for $x \in (0, \infty)$ is convex.
5. Given a convex function f and a real-valued random variable U taking values in the domain of f, we write $\mathbb{E}[f(U)]$ for the expectation of $f(U)$. What is the inequality between $\mathbb{E}[f(U)]$ and $f(\mathbb{E}[U])$ that Jensen's inequality provides?
6. Prove that the mutual information $I(X; Y) := H(X)+H(Y)-H(X, Y)$ is symmetric and non-negative. *hint*: you may use your answers to questions 4 and 5.
7. Prove that $H(X, Y) \geq 2\min(H(X), H(Y))$ and $H(X, Y) \leq 2\max(H(X), H(Y))$.

Suppose now that Y is a deterministic function of X, $Y = g(X)$, where $g : \mathcal{X} \to \mathcal{X}$. Answer the questions below, either with an example or a proof that no such examples can exist.

8. Can you find an example of a function g such that $H(Y) \leq H(X)$?
9. Can you find an example of a function g such that $H(Y) = H(X)$?
10. Can you find an example of a function g such that $H(Y) > H(X)$?

Q. 2 Given a source taking values randomly (independently, identically distributed) in a finite alphabet, Huffman proposed a lossless compression algorithm in 1952—known as Huffman coding—which creates a variable length prefix code for the symbols in that alphabet.

1. What is the definition of a variable-length prefix code?
2. Which of the following sets of codewords could be the Huffman code for some 4 symbol source alphabet? Justify your answer.
   (a) 01, 10, 00, 111;
   (b) 0, 10, 110, 111;
   (c) 1, 01, 10, 001;
   (d) 0, 110, 111, 101.